On the compositionality of monads via weak distributive laws

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Under the supervision of Marc Aiguier and Daniela Petrişan

Context

Selected contributions

- The law $DP \to PD$
- Generalised determinisation
- Compact Hausdorff spaces and the law $VV \to VV$

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Abstracting computer science $\rightarrow$ category theory

- Principle of compositionality
  - *The whole is determined by the parts and the arrangement rules*
  - Complex software is made of small programs

Abstracting computer science $\rightarrow$ category theory

- Principle of compositionality
  - *The whole is determined by the parts and the arrangement rules*
  - Complex software is made of small programs
- Category theory is relevant to computer science
  - Based on $\circ$ operator $\rightarrow$ compositional by essence
  - High abstraction $\rightarrow$ high generality
  - Behavioural $\rightarrow$ heuristics to find meaningful constructions

Effects

- Branching behaviour of a program

```
def division(p,q):
   if q == 0:
       return None
   else:
       return p//q
```

- This program outputs $\boxed{\text{some } \texttt{nat}\text{, or nothing}}$

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- This program outputs $\boxed{\texttt{Maybe nat}}$
- Monads model computational effects (Moggi 91, Plotkin - Power 02)
  e.g. Haskell language (Wadler 95)

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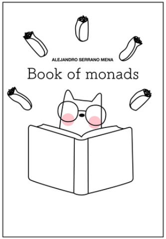
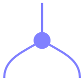


Monads

- Monads T, S, $\ldots$ are triples

functor
effect

unit
create effect



multiplication
collapse effects

Monads

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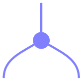

multiplication
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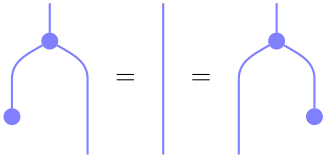
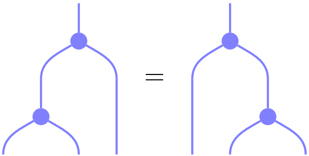
obeying 3 coherence axioms





The powerset monad P

$\{x\} \in PX$ $\qquad$ $\bigcup \mathcal{U} \in PX$

$PX = \boxed{\text{subsets of } X}$ $\qquad$ $x \in X$ $\qquad$ $\mathcal{U} \in PPX$

*technically *finite* powerset monad here

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P powerset monad* $=$ nondeterministic choice $\vee$ $=$ sup-semilattices

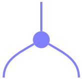
- $x \vee x = x$
- $x \vee y = y \vee x$
- $x \vee (y \vee z) = (x \vee y) \vee z$

*technically *finite* powerset monad here

The distribution monad D

Dirac
$\delta_x \in DX$

'expected value'
(mixture of Φ) $\in DX$



$x \in X$

$\Phi \in DDX$

$DX = \boxed{\text{probability distributions over } X}$

The distribution monad D

Dirac $\delta_x \in DX$ | 'expected value' (mixture of Φ) $\in DX$

$DX = \boxed{\text{probability distributions over } X}$ | $x \in X$ | $\Phi \in DDX$

D distribution monad $=$ probabilistic choice $\oplus_p$ $=$ convex algebras

- $x \oplus_1 y = x$
- $x \oplus_p x = x$
- $x \oplus_p y = y \oplus_{1-p} x$
- $(x \oplus_p y) \oplus_r z = x \oplus_{pr} \left(y \oplus_{\frac{r-pr}{1-pr}} z\right)$ if $p, r \neq 1$

Combining effects

What about composition of effects?

- $PP =$ two non-deterministic choices in a row
- $PD =$ one nondeterministic choice, then one probabilistic choice
- $DP =$ one probabilistic choice, then one nondeterministic choice
- $DD =$ two probabilistic choices in a row

Combining effects

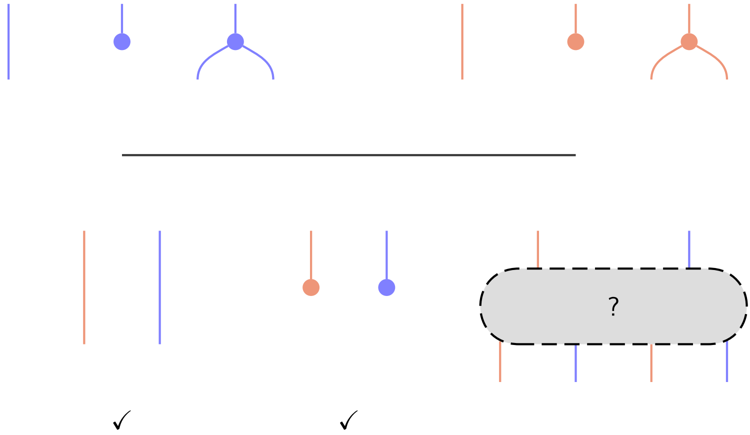
What about composition of effects?

- PP = two non-deterministic choices in a row
- PD = one nondeterministic choice, then one probabilistic choice
- DP = one probabilistic choice, then one nondeterministic choice
- DD = two probabilistic choices in a row

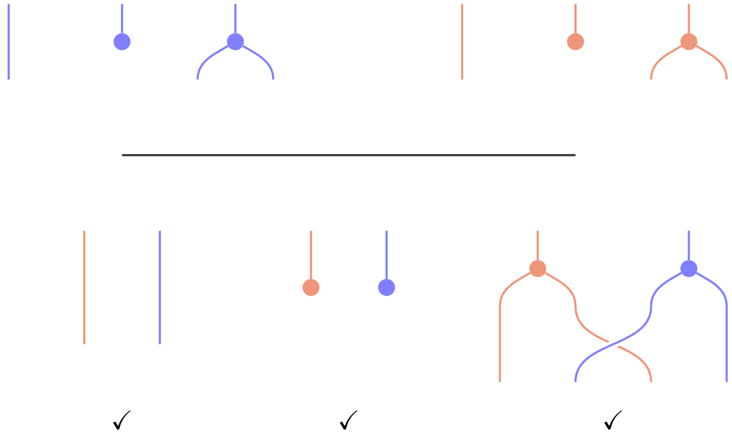
Monads do not compose in general!

- S monad $+$ T monad $\not\Rightarrow$ ST monad

Combining effects



Combining effects



Distributive laws

(Beck 69)

- A distributive law $\lambda : TS \to ST$ is a

swap effects

obeying 4 compatibility axioms

 compatible with 

 compatible with 

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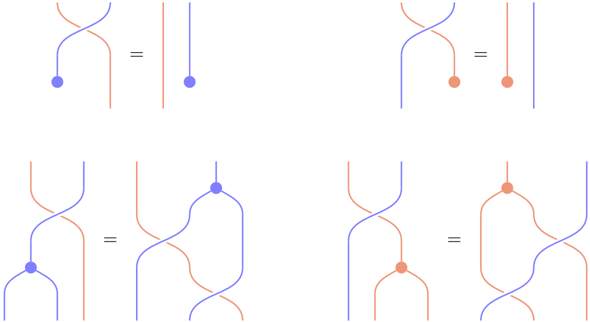

Distributive laws

$$
\begin{tikzcd}
\begin{array}{c}\text{extensions of } T\\ \text{to free } S\text{-algebras}\end{array} \arrow[rr, Leftrightarrow] \arrow[dr, Leftrightarrow] \arrow[ddr, dotted] & & \begin{array}{c}\text{liftings of } S\\ \text{to } T\text{-algebras}\end{array} \arrow[dl, Leftrightarrow] \arrow[ddl, dotted] \\
& \begin{array}{c}\text{distributive laws}\\ TS \to ST\end{array} \arrow[d, dotted] & \\
& \begin{array}{c}\text{monad structures}\\ \text{on } ST\end{array} &
\end{tikzcd}
$$

No-go theorems

- No $\lambda : DP \to PD$ (Plotkin, Varacca 03, Varacca - Winskel 06)
- No $\lambda : PP \to PP$ (Klin - Salamanca 18)
- No $\lambda : PD \to DP$ (Varacca 03, Zwart - Marsden 19)
- No $\lambda : DD \to DD$ (Zwart - Marsden 19)
- and many other no-go situations (Zwart - Marsden 19, Zwart 20)

No-go theorems

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- No $\lambda : DD \to DD$ (Zwart - Marsden 19)
- and many other no-go situations (Zwart - Marsden 19, Zwart 20)

- No monad PP (Klin - Salamanca 18)
- No monad PD (Dahlqvist - Neves 18)

Weak distributive laws

(Garner 20)

- A weak distributive law $\lambda : TS \to ST$ is a

swap effects

obeying 3 compatibility axioms

compatible with

compatible with

compatible with

Weak distributive laws

$$
\begin{tikzcd}[column sep=small]
\begin{array}{c}\text{weak}\\ \text{extensions of } T\\ \text{to free } S\text{-algebras}\end{array} \arrow[rr, Leftrightarrow, "*"] \arrow[dr, Leftrightarrow] \arrow[ddr, dotted, "\times"'] & & \begin{array}{c}\text{weak}\\ \text{liftings of } S\\ \text{to } T\text{-algebras}\end{array} \arrow[dl, Leftrightarrow, "*"'] \arrow[ddl, dotted] \\
& \begin{array}{c}\text{weak}\\ \text{distributive laws}\\ TS \to ST\end{array} \arrow[d, dotted, "\times"] & \\
& \begin{array}{c}\text{monad structures}\\ \text{almost on } ST\end{array} &
\end{tikzcd}
$$

$^*\Rightarrow$ if idempotents split in the base category

Monotone (weak) distributive laws

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$$\begin{tikzcd}
\text{extensions of } T \text{ to free } P\text{-algebras} = \text{to relations} \arrow[rr, Leftrightarrow] \arrow[dr, Leftrightarrow] \arrow[ddr, dotted, tail] & & \text{liftings of } P \text{ to } T\text{-algebras} \arrow[dl, Leftrightarrow] \arrow[ddl, dotted, tail] \\
& \text{distributive laws } TP \to PT \arrow[d, dotted, tail] & \\
& \text{monad structures on } PT &
\end{tikzcd}$$

- Monotone = the extension preserves relation inclusion
  $\approx$ well-behaved

Monotone (weak) distributive laws

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- Monotone = the extension preserves relation inclusion
  ≈ well-behaved

Theorem (Barr 70)

- There is at most one monotone distributive law $TP \to PT$.
- Existence $\iff$ T functor, unit, multiplication are weakly cartesian
- Explicit formula

Monotone (weak) distributive laws

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Theorem (Barr 70, Garner 20)

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- Existence $\iff$ T functor, multiplication are weakly cartesian
- Explicit formula

Context

Selected contributions

The law $DP \to PD$

Generalised determinisation

Compact Hausdorff spaces and the law $VV \to VV$

Contributions

- Theory
  - coweak distributive laws
  - trivial (co)weak distributive laws
  - iterated (co)weak distributive laws
- Case studies in Set
  - $DP \to PD$ and the convex powerset monad *LICS'20*
  - algebraic distributivity of $\oplus_p$ over $\vee$ *LICS'20*
  - discussion on $PD \to DP$
- Coalgebras
  - generalised determinisation of coalgebras, e.g.

    probabilistic automata via $DP \to PD$ *LICS'20*

    alternating automata via $PP \to PP$ *ICALP'21*
  - compatibility of up-to techniques
- Case studies outside Set
  - toposes, e.g. $\exists\exists \to \exists\exists$ + Coq proofs *ICALP'21*
  - compact Hausdorff spaces, e.g. $VV \to VV$ *ICALP'21*

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The law $DP \to PD$

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Distribution weakly distributes over powerset

Theorem (G. - Petrișan LICS'20)

There is a unique monotone weak distributive law $\lambda : DP \to PD$.

$$\lambda_X \left(\sum p_i \cdot U_i \right) = \left\{ \sum p_i \cdot \varphi_i | \varphi_i \text{ distribution on } U_i \right\}$$

- Requires a new technical result: D multiplication is weakly cartesian
- Works with finite distributions and countable distributions
- Provides a new categorical answer to the longstanding problem of composing probability and non-determinism:

(Mislove 00)

(Tix - Keimel - Plotkin 09)

(Keimel - Plotkin 17)

The convex powerset monad

Theorem (G. - Petrișan LICS'20)

The weak lifting corresponding to the monotone $DP \to PD$ is the convex powerset monad on convex algebras

$$(X, \oplus_p) \mapsto (\text{convex subsets of } X, \text{'pointwise' } \oplus_p)$$

i.e.

$$\begin{aligned} U \oplus_p V &= \{u \oplus_p v \mid u \in U, v \in V\} \\ U \oplus_1 V &= U \\ U \oplus_0 V &= V \end{aligned}$$

- A known monad whose existence was puzzling

  (Jacobs 08, Bonchi - Silva - Sokolova 17)
- Now obtained 'for free' via a generic procedure

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Coalgebra + distributive law → determinisation

Step 1. Standard determinisation algorithm, state space $X \mapsto PX$

$$\begin{tikzcd} x \arrow[loop above, "{a,b}"] \arrow[r, "a"] & \underline{y} \end{tikzcd} \qquad \mapsto \qquad \begin{tikzcd} \{x\} \arrow[loop above, "b"] \arrow[r, shift left, "a"] & \underline{\{x, y\}} \arrow[l, shift left, "b"] \arrow[loop above, "a"] \end{tikzcd}$$

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$$\mathrm{Coalg}(GP) \to \mathrm{Coalg}(G)$$

relying on a distributive law $PG \to GP$, where $G = 2 \times (-)^A$

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Step 3. Any distributive law $TF \to FT$ yields a generalised determinisation

$$\begin{tikzcd} \mathrm{Coalg}(FT) \arrow[r, "\text{determinisation}"'] \arrow[d, "\text{get states}"'] & \mathrm{Coalg}(F) \arrow[d, "\text{get states}"] \\ Set \arrow[r, "T", "\text{state space expansion}"'] & Set \end{tikzcd}$$

that factors through $\mathrm{Coalg}(\overline{F})$, where $\overline{F}$ is the lifting.

(Jacobs - Silva - Sokolova 15)

Coalgebra + weak distributive law → determinisation

Theorem (G. - Petrișan LICS'20)

Any weak distributive law $TF \to FT$ yields a generalised determinisation

$$\begin{tikzcd}
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\mathit{Set} \arrow[r, "T", "\text{state space expansion}"'] & \mathit{Set}
\end{tikzcd}$$

that factors through $\mathrm{Coalg}(\overline{F})$, where $\overline{F}$ is the weak lifting.

From probabilistic automata to belief-state transformers

- What gives the monotone $DP \to PD$?
- $\mathrm{Coalg}(PD)$ with states $X \approx$ probabilistic automata

  one nondeterministic choice, then one probabilistic choice
- $\mathrm{Coalg}(P)$ with states $DX \approx$ belief-state transformers

  one nondeterministic choice, states are distributions

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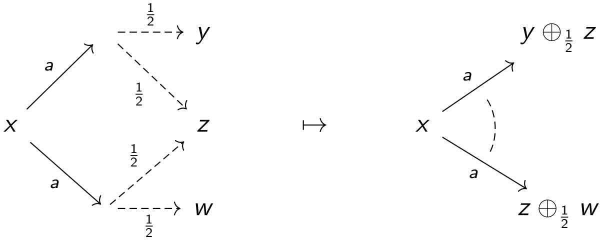

On the right, x can a-transition to any distribution $(y \oplus_{\frac{1}{2}} z) \oplus_p (z \oplus_{\frac{1}{2}} w)$.

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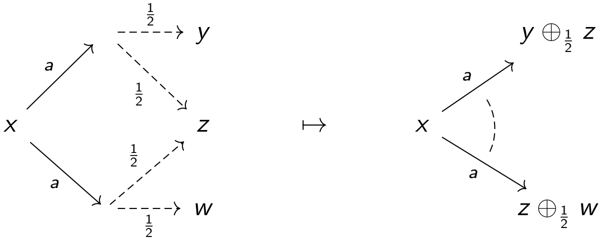

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A case study of non-Set laws

- Can we generalise $DP \to PD$ to continuous probability?
- What are advantages of categorical methods over algebraic ones?

(Parlant 20, Zwart 20)

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- Study laws in other categories than Set
- Category of compact Hausdorff spaces is convenient:

effect \category	Set	KHaus
non-determinism	powerset P	Vietoris V
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A case study of non-Set laws

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effect \category	Set	KHaus
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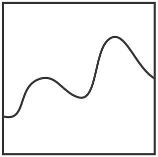
- First goal: find a Barr-like theorem
- Vietoris monad on a compact Hausdorff space X:

$\{x\} \in VX$ $\quad\quad$ $\bigcup \mathcal{U} \in VX$

$VX = \boxed{\text{closed subsets of } X}$ $\quad\quad$ $x \in X$ $\quad\quad$ $\mathcal{U} \in VVX$

Relations in KHaus

continuous functions		continuous relations		closed relations
	$\subseteq$		$\subseteq$	

(Bezhanishvili *et al.* 19)

$$\begin{tikzcd}
\mathsf{KHaus} \arrow[r, "\text{graph}"] \arrow[rr, bend right, "\text{graph}"'] & \text{free } V\text{-algebras} \arrow[r, "\text{forget continuity}"] & \mathsf{Rel}(\mathsf{KHaus})
\end{tikzcd}$$

Weak distributive laws in KHaus

Theorem (G. - Petrișan - Aiguier ICALP'21)

- There is at most one* monotone distributive law $TV \to VT$.
- Existence $\iff$ T functor, unit, multiplication are nearly cartesian

  T preserves strong epis and Rel(T) preserves continuity

*at most one coming from a relational extension

Weak distributive laws in KHaus

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Theorem (G. - Petrişan - Aiguier ICALP'21)

There is a monotone weak distributive law $VV \to VV$. For $\mathcal{C} \in VVX$,

$$\lambda_X(\mathcal{C}) = \left(\Box \bigcup_{C \in \mathcal{C}} C \right) \cap \left(\bigcap_{C \in \mathcal{C}} \Diamond C \right)$$

where

$$\Box C = \{B \text{ closed in } X \mid B \subseteq C\}$$
$$\Diamond C = \{B \text{ closed in } X \mid B \cap C \neq \emptyset\}$$

*at most one coming from a relational extension

Conclusion

Weak distributive laws are relevant to tackle 'almost working' cases

1. Finally combines probabilistic choice and nondeterministic choice, categorically
2. Explains mysterious results from the literature
3. More versatile than algebraic methods, KHaus as a proof of concept

Future work

Conjecture (Generalised $DP \to PD$)

There is a monotone weak distributive law $RV \to VR$. For $m \in RVX$,

$$\lambda_X(m) = \left\{ \begin{array}{l} m' \text{ Radon measure on } X \text{ such that} \\ \forall(\mathcal{C}, B) \in VVX \times VX, \bigcup \mathcal{C} \subseteq B \Rightarrow m(\mathcal{C}) \leq m'(B) \end{array} \right\}$$

- Other laws : are there
  - Non-trivial coweak distributive laws?
  - Non-trivial non-monotone weak distributive laws?
  - No-go results e.g. $PD \to DP$?
  - Meaningful laws in other categories e.g. quasi-Borel spaces?

Thank you!

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Coweak distributive laws

- A coweak distributive law $\lambda : TS \to ST$ is a

swap effects

obeying 3 compatibility axioms

 compatible with 

 compatible with 

 compatible with 

- coweak distributive law $\Leftrightarrow$ coweak lifting $\Leftrightarrow^*$ coweak extension

  $\Rightarrow$ monad almost on ST

---

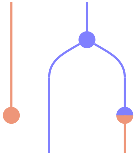
*$\Rightarrow$ if every retract of a free S-algebra is free

Trivial weak distributive laws

- A monad morphism is a  obeying 2 obvious compatibility axioms

Theorem (G.)

A monad morphism yields a weak distributive law  defined by



but the composite monad is just the blue one.

- Example: trivial weak distributive law $PP \to PP$

$$\lambda_X(\mathcal{U}) = \left\{ \bigcup \mathcal{U} \right\}$$

Weak extension on Rel is the 'relation graph' functor

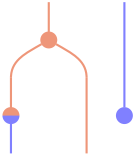
$$R \subseteq X \times Y \mapsto \{(U, R[U]) \mid U \subseteq X\} \subseteq PX \times PY$$

Trivial coweak distributive laws

- A monad morphism is a  obeying 2 obvious compatibility axioms

Theorem (G.)

A monad morphism yields a coweak distributive law  defined by



but the composite monad is just the orange one.

- Example: trivial coweak distributive law $PP \to PP$

$$\lambda_X(\mathcal{U}) = \left\{ \{x\} \mid x \in \bigcup \mathcal{U} \right\}$$

  Coweak lifting on sup-semilattices is the 'make free' functor

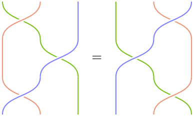
$$(X, \vee) \mapsto \left(PX, \bigcup \right)$$

Iterated (co)weak distributive laws

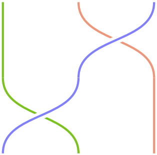
Given monads

and distributive laws

such that the Yang-Baxter equation holds

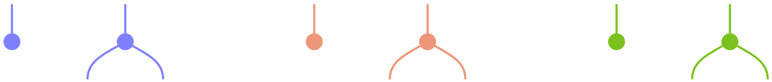


then the following is a distributive law (Cheng 11)



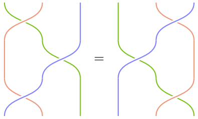
Iterated (co)weak distributive laws

Given monads

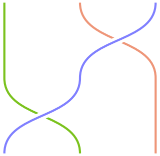


and distributive laws (weak - plain - weak)

such that the Yang-Baxter equation holds



then the following is a weak distributive law (Cheng 11)



Algebraic distributivity of $\oplus_p$ over $\vee$

Theorem (Bonchi - Sokolova - Vignudelli 19)

The *monad of convex, non-empty, finitely generated subsets of distributions* on Set is presented by the theory of *convex semilattices* i.e.

- theory of sup-semilattices $\vee$
- theory of convex algebras $\oplus_p$
- distributivity axiom

$$(x \vee y) \oplus_p z = (x \oplus_p z) \vee (y \oplus_p z)$$

Let $P_c D$ be the monad of convex subsets of distributions on Set. The monotone weak distributive law $\lambda : DP \to PD$ and the fact

$$\lambda\text{-algebras} \cong P_c D\text{-algebras}$$

yield a similar result, with infinite distributivity

$$\left(\bigvee x_i\right) \oplus_p z = \bigvee (x_i \oplus_p z)$$

Discussion on $PD \to DP$

There is probably no meaningful weak distributive law $PD \to DP$

- There is no such distributive law.
- Imposing distributivity

$$x \vee (y \oplus_p z) = (x \vee y) \oplus_p (x \vee z)$$

leads to no quantitative content (Keimel - Plotkin 17)

A new argument is

Theorem (G.)

Even with finite P's, there is no natural transformation α such that

$$\begin{array}{ccc} PD & \xrightarrow{\alpha} & DP \\ {\scriptstyle P\mathrm{supp}}\downarrow & & \downarrow{\scriptstyle \mathrm{supp}P} \\ PP & \xrightarrow[\lambda]{} & PP \end{array}$$

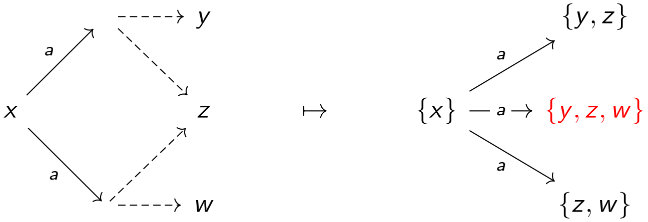
where λ is the monotone weak distributive law $PP \to PP$.

Future work: prove a no-go theorem.

Generalised determinisation of alternating automata

sort of law	$\lambda_X(\mathcal{U})$ formula	$\bigvee\bigwedge\mathcal{U}$ CNF
trivial weak	$\bigcup\mathcal{U}$	$\bigwedge\bigwedge\mathcal{U}$ DNF
trivial coweak	$\{\{x\} \mid x \in \bigcup\mathcal{U}\}$	$\bigvee\bigvee\mathcal{U}$ DNF
monotone weak	$(\Box\bigcup_{U\in\mathcal{U}} U) \cap (\bigcap_{U\in\mathcal{U}} \Diamond U)$	$\bigvee\bigwedge\mathcal{U}$ DNF

- Using the monotone weak distributive law $PP \to PP$:



- Explains a semantically correct determinisation from (Klin - Rot 16)

Compatibility of up-to techniques

Theorem (G.)

For an algebraic expansion due to a weak distributive law $\lambda : TF \to FT$

- context is a compatible up-to technique
- congruence is a compatible up-to technique (if F weakly cartesian)

i.e. one can compute bisimulations up to $\equiv$

- Accelerates computations of bisimulations
- Specific to weak laws: erases 'additional' states due to weakness
- Explains bisimulation up-to convex hull for probabilistic automata

(Bonchi - Silva - Sokolova 17)

Toposes and $\exists\exists \to \exists\exists$

Fact: every topos has a powerset monad $\exists$.

Theorem (G. - Petrișan - Aiguier ICALP'21)

- There is at most one monotone distributive law $T\exists \to \exists T$.
- Existence $\iff$ T functor, unit, multiplication are nearly cartesian

  T preserves strong epis

Toposes and $\exists\exists \to \exists\exists$

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Theorem (G. - Petrișan - Aiguier ICALP'21)

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Toposes and $\exists\exists \to \exists\exists$

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- There is at most one monotone weak distributive law $T\exists \to \exists T$.
- Existence $\iff$ T functor, multiplication are nearly cartesian

  T preserves strong epis

Theorem (G. - Petrișan - Aiguier ICALP'21)

There is a monotone weak distributive law $\exists\exists \to \exists\exists$. In the internal logic,

$$(t : \Omega^{\Omega^X}) \vdash \lambda_X(t) = \{s : \Omega^X \mid (\forall(x : X), x \in s \to x \in \mu^{\exists}_X(t) \\ \wedge \forall(u : \Omega^X).u \in t \to \exists(x : X).x \in u \wedge x \in s\}$$

This is a distributive law iff the topos is degenerate.

- Generalisation of the monotone $PP \to PP$ in Set
- Intermediate result before KHaus

Coq proofs for toposes

- Proofs in Coq $\equiv$ proofs in the internal logic
- Prop $\equiv$ subobject classifier

```
Definition P X := X → Prop.
Definition im [X] [Y] (f : X → Y) (a : P X) (y : Y) :=
   exists (x : X), a x ∧ f x = y.
Definition etaP X (x : X) (x' : X) := x = x'.
Definition muP X (t : P (P X)) (x : X) := exists (s : P X), s x ∧ t s.
```

```
Theorem eta_nearly_cartesian :
  (forall X Y (f : X → Y) (s : P X) (y : Y),
  im f s = etaP Y y →
  exists (x : X), etaP X x = s ∧ f x = y) ↔ (True = False).
```

```
Theorem mu_nearly_cartesian :
  forall X Y (f : X → Y) (s : P X) (t' : P (P Y)),
  im f s = muP Y t' →
  exists (t : P (P X)), muP X t = s ∧ im (im f) t = t'.
```

```
Theorem monotone_weak_dlaw X : forall (t : P (P X)) (s : P X),
   lambda_m X t s ↔
   (forall x : X, s x → muP X t x)
    ∧ (forall u : (P X), t u → exists (x : X), u x ∧ s x).
```

- ... and a formalisation of No $PP \to PP$ (Klin - Salamanca 18)

The conjecture $RV \to VR$

Conjecture (Generalised $DP \to PD$)

There is a monotone weak distributive law $RV \to VR$. For $m \in RVX$,

$$\lambda_X(m) = \left\{ \begin{array}{l} m' \text{ Radon measure on } X \text{ such that} \\ \forall(\mathcal{C}, B) \in VVX \times VX, \bigcup \mathcal{C} \subseteq B \Rightarrow m(\mathcal{C}) \leq m'(B) \end{array} \right\}$$

Proof.

- R preserves strong epis ✓
- R is nearly cartesian ✓
- multiplication is nearly cartesian ?
- $\mathrm{Rel}(R)$ preserves continuity ?

Expression of λ is obtained via (Edwards 78). □

Citations

The slides cite [16, 18, 22, 2, 20, 21, 14, 24, 23, 8, 1, 9, 10, 15, 19, 13, 11, 4, 12, 3, 5, 7, 17, 6]