On the compositionality of monads via weak distributive laws

Alexandre Goy

MICS, CentraleSupélec, Université Paris-Saclay

PhD defence - 19 October 2021

Under the supervision of Marc Aiguier and Daniela Petrișan
Context

Selected contributions
The law $DP \rightarrow PD$
Generalised determinisation
Compact Hausdorff spaces and the law $VV \rightarrow VV$
Context

Selected contributions
The law $DP \rightarrow PD$
Generalised determinisation
Compact Hausdorff spaces and the law $VV \rightarrow VV$
Abstracting computer science → category theory

- Principle of compositionality
  - *The whole is determined by the parts and the arrangement rules*
  - Complex software is made of small programs
Abstracting computer science → category theory

- Principle of compositionality
  - *The whole is determined by the parts and the arrangement rules*
  - Complex software is made of small programs

- Category theory is relevant to computer science
  - Based on \( \circ \) operator → compositional by essence
  - High abstraction → high generality
  - Behavioural → heuristics to find meaningful constructions
Effects

▶ Branching behaviour of a program

def division(p,q):
    if q == 0:
        return None
    else:
        return p//q

▶ This program outputs some nat, or nothing
Effects

- Branching behaviour of a program

```python
def division(p, q):
    if q == 0:
        return None
    else:
        return p // q
```

- This program outputs some `nat`, or nothing
- This program outputs `Maybe nat`
- **Monads** model computational effects (Moggi 91, Plotkin - Power 02) e.g. Haskell language (Wadler 95)
Effects

- Branching behaviour of a program

```python
def division(p,q):
    if q == 0:
        return None
    else:
        return p//q
```

- This program outputs \texttt{some nat, or nothing}
- This program outputs \texttt{Maybe nat}
- 
  \textbf{Monads} model computational effects (Moggi 91, Plotkin - Power 02)

  e.g. Haskell language

\textit{Book of monads}
Monads

Monads $T, S, \ldots$ are triples

- functor
- effect
- unit
- create effect
- multiplication
- collapse effects
Monads

Monads $T, S, \ldots$ are triples

- functor
- unit
- multiplication

obeying 3 coherence axioms
The powerset monad $P$

\[ PX = \{ \text{subsets of } X \} \]

\[ \{ x \} \in PX \]

\[ \bigcup \mathcal{U} \in PX \]

\[ x \in X \]

\[ \mathcal{U} \in PPX \]

*technically finite powerset monad here*
The powerset monad $P$

$\{x\} \in PX$ \hspace{2cm} $\bigcup \mathcal{U} \in PX$

$PX = \text{subsets of } X$ \hspace{2cm} $x \in X$ \hspace{2cm} $\mathcal{U} \in PPX$

$P$ powerset monad* = nondeterministic choice $\lor = \text{sup-semilattices}$

- $x \lor x = x$
- $x \lor y = y \lor x$
- $x \lor (y \lor z) = (x \lor y) \lor z$

*technically *finite* powerset monad here
The distribution monad $D$

$DX = \text{probability distributions over } X$

Dirac
$\delta_x \in DX$

'expected value'
$(\text{mixture of } \Phi) \in DX$

$x \in X$

$\Phi \in DDX$
The distribution monad $D$

$D$ distribution monad = probabilistic choice $\oplus_p = \text{convex algebras}$

- $x \oplus_1 y = x$
- $x \oplus_p x = x$
- $x \oplus_p y = y \oplus_{1-p} x$
- $(x \oplus_p y) \oplus_r z = x \oplus_{pr} \left(y \oplus_{\frac{r-pr}{1-pr}} z\right)$ if $p, r \neq 1$
Combining effects

What about composition of effects?

- $PP = \text{two non-deterministic choices in a row}$
- $PD = \text{one nondeterministic choice, then one probabilistic choice}$
- $DP = \text{one probabilistic choice, then one nondeterministic choice}$
- $DD = \text{two probabilistic choices in a row}$
Combining effects

What about composition of effects?

- **$PP$** = two non-deterministic choices in a row
- **$PD$** = one nondeterministic choice, then one probabilistic choice
- **$DP$** = one probabilistic choice, then one nondeterministic choice
- **$DD$** = two probabilistic choices in a row

Monads do not compose in general!

- **$S$ monad $+$ $T$ monad $\not\Rightarrow ST$ monad**
Combining effects
Distributive laws

A distributive law $\lambda : TS \to ST$ is a

swap effects

obeying 4 compatibility axioms

\[ \text{compatible with} \]

\[ \text{compatible with} \]

\[ \text{compatible with} \]

\[ \text{compatible with} \]
A distributive law $\lambda : TS \to ST$ is a swap effects obeying 4 compatibility axioms.

(beck 69)
Distributive laws

extensions of $T$ to free $S$-algebras

liftings of $S$ to $T$-algebras

distributive laws

$TS \rightarrow ST$

monad structures on $ST$
No-go theorems

- No $\lambda : DP \rightarrow PD$ (Plotkin, Varacca 03, Varacca - Winskel 06)
- No $\lambda : PP \rightarrow PP$ (Klin - Salamanca 18)
- No $\lambda : PD \rightarrow DP$ (Varacca 03, Zwart - Marsden 19)
- No $\lambda : DD \rightarrow DD$ (Zwart - Marsden 19)
- and many other no-go situations (Zwart - Marsden 19, Zwart 20)
No-go theorems

- No $\lambda : DP \rightarrow PD$ (Plotkin, Varacca 03, Varacca - Winskel 06)
- No $\lambda : PP \rightarrow PP$ (Klin - Salamanca 18)
- No $\lambda : PD \rightarrow DP$ (Varacca 03, Zwart - Marsden 19)
- No $\lambda : DD \rightarrow DD$ (Zwart - Marsden 19)
- and many other no-go situations (Zwart - Marsden 19, Zwart 20)

- No monad $PP$ (Klin - Salamanca 18)
- No monad $PD$ (Dahlqvist - Neves 18)
Weak distributive laws

A weak distributive law $\lambda : TS \rightarrow ST$ is a swap effects obeying 3 compatibility axioms.
Weak distributive laws

weak extensions of $T$ to free $S$-algebras

weak distributive laws $TS \to ST$

weak liftings of $S$ to $T$-algebras

monad structures almost on $ST$

* if idempotents split in the base category
Monotone (weak) distributive laws
Monotone (weak) distributive laws

extensions of $T$
to free $P$-algebras
$\approx$ to relations

liftings of $P$
to $T$-algebras

distributive laws
$TP \to PT$

monad structures
on $PT$

$\blacktriangleright$ Monotone $\equiv$ the extension preserves relation inclusion
$\approx$ well-behaved
Monotone (weak) distributive laws

extensions of $T$

to free $P$-algebras

$\Rightarrow$ to relations

liftings of $P$

to $T$-algebras

\[ TP \rightarrow PT \]

monad structures

on $PT$

$\Rightarrow$ Monotone $=$ the extension preserves relation inclusion

$\Rightarrow$ well-behaved

Theorem (Barr 70)

$\Rightarrow$ There is at most one monotone distributive law $TP \rightarrow PT$.

$\Rightarrow$ Existence $\iff T$ functor, unit, multiplication are weakly cartesian

$\Rightarrow$ Explicit formula
Monotone (weak) distributive laws

Extensions of $T$ to free $P$-algebras = to relations

Liftings of $P$ to $T$-algebras

Distributive laws $TP \to PT$

Monad structures on $PT$

Monotone = the extension preserves relation inclusion ≈ well-behaved

Theorem (Barr 70, Garner 20)

There is at most one monotone weak distributive law $TP \to PT$.

Existence $\iff T$ functor, multiplication are weakly cartesian

Explicit formula
Selected contributions

The law $DP \rightarrow PD$

Generalised determinisation

Compact Hausdorff spaces and the law $VV \rightarrow VV$
Contributions

▶ Theory
▶ coweak distributive laws
▶ trivial (co)weak distributive laws
▶ iterated (co)weak distributive laws

▶ Case studies in Set
▶ $DP \to PD$ and the convex powerset monad  \textit{LICS'20}
▶ algebraic distributivity of $\bigoplus_p$ over $\lor$  \textit{LICS'20}
▶ discussion on $PD \to DP$

▶ Coalgebras
▶ generalised determinisation of coalgebras, e.g.
  probabilistic automata via $DP \to PD$  \textit{LICS'20}
  alternating automata via $PP \to PP$  \textit{ICALP'21}
▶ compatibility of up-to techniques

▶ Case studies outside Set
▶ toposes, e.g. $\exists \exists \to \exists \exists +$ Coq proofs  \textit{ICALP'21}
▶ compact Hausdorff spaces, e.g. $VV \to VV$  \textit{ICALP'21}
Context

Selected contributions

The law $DP \rightarrow PD$

Generalised determinisation

Compact Hausdorff spaces and the law $VV \rightarrow VV$
Distribution weakly distributes over powerset

**Theorem (G. - Petrișan LICS’20)**

There is a unique monotone weak distributive law \( \lambda : DP \rightarrow PD \).

\[
\lambda_X \left( \sum p_i \cdot U_i \right) = \left\{ \sum p_i \cdot \varphi_i \mid \varphi_i \text{ distribution on } U_i \right\}
\]

- Requires a new technical result: \( D \) multiplication is weakly cartesian
- Works with finite distributions and countable distributions
- Provides a new categorical answer to the longstanding problem of composing probability and non-determinism:

  *(Mislove 00)*

  *(Tix - Keimel - Plotkin 09)*

  *(Keimel - Plotkin 17)*
The convex powerset monad

Theorem (G. - Petrişan LICS’20)
The weak lifting corresponding to the monotone $DP \to PD$ is the convex powerset monad on convex algebras

$$(X, \oplus_p) \mapsto (\text{convex subsets of } X, \text{'}pointwise’ \oplus_p)$$
i.e.

$$U \oplus_p V = \{ u \oplus_p v \mid u \in U, v \in V \}$$
$$U \oplus_1 V = U$$
$$U \oplus_0 V = V$$

- A known monad whose existence was puzzling
  (Jacobs 08, Bonchi - Silva - Sokolova 17)
- Now obtained ’for free’ via a generic procedure
Context

Selected contributions

The law $DP \rightarrow PD$

Generalised determinisation

Compact Hausdorff spaces and the law $VV \rightarrow VV$
Coalgebra + distributive law $\rightarrow$ determinisation

**Step 1.** Standard determinisation algorithm, state space $X \mapsto P X$

\[
\begin{align*}
\text{x} \xrightarrow{a,b} \text{y} & \quad \mapsto \quad \{x, y\} &\quad \begin{array}{c}
\text{x} \xrightarrow{b} \{x\} & \quad \begin{array}{c}
\text{x} \xrightarrow{a} \{x, y\}
\end{array}
\end{array}
\end{align*}
\]
Coalgebra + distributive law $\rightarrow$ determinisation

Step 1. Standard determinisation algorithm, state space $X \mapsto PX$

$\begin{array}{c}
a, b \\
\circlearrowleft \\
x \\
\xrightarrow{a} \\
y \\
\mapsto \\
\{x\} \\
\xleftarrow{b} \\
\{x, y\}
\end{array}$

Step 2. Determinisation is a functor between categories of coalgebras

$$\text{Coalg}(GP) \rightarrow \text{Coalg}(G)$$

relying on a distributive law $PG \rightarrow GP$, where $G = 2 \times (-)^A$
Coalgebra + distributive law $\rightarrow$ determinisation

**Step 1.** Standard determinisation algorithm, state space $X \mapsto PX$

\[
\begin{array}{c}
\xrightarrow{a,b} \quad x \xrightarrow{a} y \\
\end{array}
\quad \implies 
\begin{array}{c}
\quad \xleftarrow{b} \quad \{x\} \xrightarrow{a} \{x, y\} \quad \xleftarrow{b}
\end{array}
\]

**Step 2.** Determinisation is a functor between categories of coalgebras

\[\text{Coalg}(GP) \rightarrow \text{Coalg}(G)\]

relying on a **distributive law** $PG \rightarrow GP$, where $G = 2 \times (-)^A$

**Step 3.** Any distributive law $TF \rightarrow FT$ yields a generalised determinisation

\[
\begin{array}{c}
\xrightarrow{\text{determinisation}} \\
\xleftarrow{\text{get states}} \\
\xrightarrow{T} \\
\xleftarrow{\text{state space expansion}}
\end{array}
\]

that factors through $\text{Coalg}(\overline{F})$, where $\overline{F}$ is the lifting.

(Jacobs - Silva - Sokolova 15)
Coalgebra + weak distributive law $\rightarrow$ determinisation

Theorem (G. - Petrişan LICS’20)

Any weak distributive law $TF \rightarrow FT$ yields a generalised determinisation

\[
\text{Coalg}(FT) \xrightarrow{\text{determinisation}} \text{Coalg}(F) \xrightarrow{\text{get states}} \text{Set} \xrightarrow{T} \text{Set} \xrightarrow{\text{get states}} \text{Coalg}(
\overline{F})
\]

that factors through $\text{Coalg}(
\overline{F})$, where $\overline{F}$ is the weak lifting.
From probabilistic automata to belief-state transformers

▶ What gives the monotone $DP \to PD$?
▶ $\text{Coalg}(PD)$ with states $X \approx$ probabilistic automata
  one nondeterministic choice, then one probabilistic choice
▶ $\text{Coalg}(P)$ with states $DX \approx$ belief-state transformers
  one nondeterministic choice, states are distributions
From probabilistic automata to belief-state transformers

▶ What gives the monotone $DP \to PD$?

▶ $\text{Coalg}(PD)$ with states $X \approx$ probabilistic automata
  one nondeterministic choice, then one probabilistic choice

▶ $\text{Coalg}(P)$ with states $DX \approx$ belief-state transformers
  one nondeterministic choice, states are distributions

On the right, $x$ can $a$-transition to any distribution $(y \oplus \frac{1}{2} z) \oplus_p (z \oplus \frac{1}{2} w)$. 
From probabilistic automata to belief-state transformers

- **What gives the monotone** $DP \rightarrow PD$?

- **$\text{Coalg}(PD)$ with states $X \approx$ probabilistic automata**
  - one nondeterministic choice, then one probabilistic choice

- **$\text{Coalg}(P)$ with states $DX \approx$ belief-state transformers**
  - one nondeterministic choice, states are distributions

On the right, $x$ can $a$-transition to any distribution $(y \oplus \frac{1}{2} z) \oplus_p (z \oplus \frac{1}{2} w)$.

- **A known determinisation whose existence was puzzling**
  (Bonchi - Silva - Sokolova 17)

- **Now obtained ’for free’ via a generic procedure**
Context

Selected contributions
The law $DP \rightarrow PD$

Generalised determinisation

Compact Hausdorff spaces and the law $VV \rightarrow VV$
A case study of non-Set laws

- Can we generalise $DP \rightarrow PD$ to continuous probability?
- What are advantages of categorical methods over algebraic ones?

(Parlant 20, Zwart 20)
A case study of non-Set laws

- Can we generalise $DP \rightarrow PD$ to continuous probability?
- What are advantages of categorical methods over algebraic ones? (Parlant 20, Zwart 20)
- Study laws in other categories than Set
- Category of compact Hausdorff spaces is convenient:

<table>
<thead>
<tr>
<th>effect \ category</th>
<th>Set</th>
<th>KHaus</th>
</tr>
</thead>
<tbody>
<tr>
<td>non-determinism</td>
<td>powerset $P$</td>
<td>Vietoris $V$</td>
</tr>
<tr>
<td>probability</td>
<td>distribution $D$</td>
<td>Radon $R$</td>
</tr>
</tbody>
</table>
A case study of non-Set laws

▸ Can we generalise $DP \to PD$ to continuous probability?
▸ What are advantages of categorical methods over algebraic ones?
  
  (Parlant 20, Zwart 20)

▸ Study laws in other categories than Set
▸ Category of compact Hausdorff spaces is convenient:

<table>
<thead>
<tr>
<th>effect \ category</th>
<th>Set</th>
<th>KHaus</th>
</tr>
</thead>
<tbody>
<tr>
<td>non-determinism probability</td>
<td>powerset $P$</td>
<td>Vietoris $V$</td>
</tr>
<tr>
<td></td>
<td>distribution $D$</td>
<td>Radon $R$</td>
</tr>
</tbody>
</table>

▸ First goal: find a Barr-like theorem
▸ Vietoris monad on a compact Hausdorff space $X$:

\[
\{x\} \in VX \quad \bigcup U \in VX
\]

\[
VX = \text{closed subsets of } X \quad x \in X \quad U \in VVX
\]
Relations in KHaus

continuous functions ⊆ continuous relations ⊆ closed relations

(Bezhanishvili et al. 19)

KHaus → free V-algebras \xrightarrow{\text{forget continuity}} \text{Rel}(KHaus)

graph
Weak distributive laws in KHaus

Theorem (G. - Petrişan - Aiguier ICALP’21)

- There is at most one* monotone distributive law $TV \rightarrow VT$.
- Existence $\iff T$ functor, unit, multiplication are nearly cartesian
  $T$ preserves strong epis and $\text{Rel}(T)$ preserves continuity

---

* at most one coming from a relational extension
Weak distributive laws in KHaus

Theorem (G. - Petrişan - Aiguier ICALP’21)

- There is at most one* monotone weak distributive law $TV \to VT$.
- Existence $\iff T$ functor, multiplication are nearly cartesian
  $T$ preserves strong epis and $\text{Rel}(T)$ preserves continuity

*at most one coming from a relational extension
Weak distributive laws in KHaus

Theorem (G. - Petrişan - Aiguier ICALP’21)

- There is at most one* monotone weak distributive law $TV \rightarrow VT$.
- Existence $\iff$ $T$ functor, multiplication are nearly cartesian $T$ preserves strong epis and $\text{Rel}(T)$ preserves continuity

Theorem (G. - Petrişan - Aiguier ICALP’21)

There is a monotone weak distributive law $VV \rightarrow VV$. For $C \in VVX$, 

$$
\lambda_X(C) = \left(\Box \bigcup_{C \in C} C\right) \cap \left(\bigcap_{C \in C} \Diamond C\right)
$$

where

$$
\Box C = \{ B \text{ closed in } X \mid B \subseteq C\}
$$

$$
\Diamond C = \{ B \text{ closed in } X \mid B \cap C \neq \emptyset\}
$$

*at most one coming from a relational extension
Weak distributive laws are relevant to tackle 'almost working' cases

1. Finally combines probabilistic choice and nondeterministic choice, categorically
2. Explains mysterious results from the literature
3. More versatile than algebraic methods, KHaus as a proof of concept
Future work

Conjecture (Generalised $DP \to PD$)

There is a monotone weak distributive law $RV \to VR$. For $m \in RV_X$,

$$\lambda_X(m) = \left\{ m' \text{ Radon measure on } X \text{ such that } \forall (C, B) \in VVX \times VX, \bigcup C \subseteq B \Rightarrow m(C) \leq m'(B) \right\}$$

- Other laws: are there
  - Non-trivial coweak distributive laws?
  - Non-trivial non-monotone weak distributive laws?
  - No-go results e.g. $PD \to DP$?
  - Meaningful laws in other categories e.g. quasi-Borel spaces?
Thank you!
References I

M. Barr.
Relational algebras.

J. Beck.
Distributive laws.

Compact Hausdorff spaces with relations and Gleason spaces.

F. Bonchi, A. Silva, and A. Sokolova.
The power of convex algebras.
References II

E. Cheng.
Iterated distributive laws.

F. Dahlqvist and R. Neves.
Compositional semantics for new paradigms: probabilistic, hybrid and beyond.

D. Edwards.
On the existence of probability measures with given marginals.

R. Garner.
The Vietoris monad and weak distributive laws.
A. Goy and D. Petrişan.
Combining probabilistic and non-deterministic choice via weak distributive laws.

A. Goy, D. Petrişan, and M. Aiguier.
Powerset-like monads weakly distribute over themselves in toposes and compact Hausdorff spaces.

B. Jacobs.
Coalgebraic trace semantics for combined possibilistic and probabilistic systems.
B. Jacobs, A. Silva, and A. Sokolova.  
Trace semantics via determinization.  
Selected Papers of CMCS’12.

K. Keimel and G. Plotkin.  
Mixed powerdomains for probability and nondeterminism.  

B. Klin and J. Salamanca.  
Iterated covariant powerset is not a monad.  

M. Mislove.  
Nondeterminism and probabilistic choice: obeying the laws.  
E. Moggi.
Notions of computation and monads.
Selected papers of LICS’89.

L. Parlant.
Monad composition via preservation of algebras.

G. Plotkin and J. Power.
Notions of computation determine monads.

Semantic domains for combining probability and non-determinism.
D. Varacca.
Probability, nondeterminism and concurrency: two denotational models for probabilistic computation.

D. Varacca and G. Winskel.
Distributing probability over non-determinism.

P. Wadler.
Monads for functional programming.

M. Zwart.
On the non-compositionality of monads via distributive laws.
A coweak distributive law $\lambda : TS \to ST$ is a swap effects obeying 3 compatibility axioms.

Coweak distributive law $\iff$ coweak lifting $\iff^* \text{coweak extension} \implies \text{monad almost on } ST$

*$\implies$ if every retract of a free $S$-algebra is free
Trivial weak distributive laws

- A monad morphism is a mapping obeying 2 obvious compatibility axioms.

**Theorem (G.)**

A monad morphism yields a weak distributive law defined by

but the composite monad is just the blue one.

- Example: trivial weak distributive law \( PP \rightarrow PP \)

\[
\lambda_X(U) = \left\{ \bigcup U \right\}
\]

Weak extension on Rel is the 'relation graph' functor

\[
R \subseteq X \times Y \mapsto \{(U, R[U]) \mid U \subseteq X\} \subseteq PX \times PY
\]
Trivial coweak distributive laws

A monad morphism is a \( \text{obeying 2 obvious compatibility axioms} \)

Theorem (G.)

A monad morphism yields a coweak distributive law defined by

but the composite monad is just the orange one.

Example: trivial coweak distributive law \( PP \to PP \)

\[
\lambda_X(U) = \left\{ \{x\} \mid x \in \bigcup U \right\}
\]

Coweak lifting on sup-semilattices is the ’make free’ functor

\[
(X, \lor) \mapsto (PX, \bigcup)
\]
Given monads

and distributive laws

such that the Yang-Baxter equation holds

then the following is a distributive law (Cheng 11)
Iterated (co)weak distributive laws

Given monads

and distributive laws (weak - plain - weak)

such that the Yang-Baxter equation holds

then the following is a weak distributive law (Cheng 11)
Algebraic distributivity of $\oplus_p$ over $\lor$

Theorem (Bonchi - Sokolova - Vignudelli 19)

The *monad of convex, non-empty, finitely generated subsets of distributions* on Set is presented by the theory of *convex semilattices* i.e.

- theory of sup-semilattices $\lor$
- theory of convex algebras $\oplus_p$
- distributivity axiom

$$(x \lor y) \oplus_p z = (x \oplus_p z) \lor (y \oplus_p z)$$

Let $P_cD$ be the monad of convex subsets of distributions on Set. The monotone weak distributive law $\lambda : DP \to PD$ and the fact

$$\lambda\text{-algebras } \cong P_cD\text{-algebras}$$

yield a similar result, with infinite distributivity

$$\left( \lor x_i \right) \oplus_p z = \lor (x_i \oplus_p z)$$
Discussion on $PD \rightarrow DP$

There is probably no meaningful weak distributive law $PD \rightarrow DP$

- There is no such distributive law.
- Imposing distributivity

\[
x \lor (y \oplus_p z) = (x \lor y) \oplus_p (x \lor z)
\]

leads to no quantitative content \hspace{1cm} (Keimel - Plotkin 17)

A new argument is

**Theorem (G.)**

Even with finite $P$'s, there is no natural transformation $\alpha$ such that

\[
\begin{array}{ccc}
PD & \xrightarrow{\alpha} & DP \\
\downarrow P_{supp} & & \downarrow \text{supp}^P \\
PP & \xrightarrow{\lambda} & PP
\end{array}
\]

where $\lambda$ is the monotone weak distributive law $PP \rightarrow PP$.

Future work: prove a no-go theorem.
Generalised determinisation of alternating automata

<table>
<thead>
<tr>
<th>sort of law</th>
<th>$\lambda_X(U)$ formula</th>
<th>$\lor \land U$ CNF</th>
</tr>
</thead>
<tbody>
<tr>
<td>trivial weak</td>
<td>$\bigcup U$</td>
<td>$\land \land U$ DNF</td>
</tr>
<tr>
<td>trivial coweak</td>
<td>${ {x} \mid x \in \bigcup U}$</td>
<td>$\lor \lor U$ DNF</td>
</tr>
<tr>
<td>monotone weak</td>
<td>$(\Box \bigcup_{U \in U} U) \cap (\bigcap_{U \in U} \Diamond U)$</td>
<td>$\lor \land U$ DNF</td>
</tr>
</tbody>
</table>

- Using the monotone weak distributive law $PP \rightarrow PP$:

- Explains a semantically correct determinisation from (Klin - Rot 16)
Compatibility of up-to techniques

Theorem (G.)

For an algebraic expansion due to a weak distributive law $\lambda : TF \to FT$

- context is a compatible up-to technique
- congruence is a compatible up-to technique (if $F$ weakly cartesian)

i.e. one can compute bisimulations up to $\equiv$

- Accelerates computations of bisimulations
- **Specific to weak laws:** erases 'additional' states due to weakness
- Explains bisimulation up-to convex hull for probabilistic automata

(Bonchi - Silva - Sokolova 17)
Toposes and $\mathbb{E} \to \mathbb{E}$

Fact: every topos has a powerset monad $\mathbb{E}$.

Theorem (G. - Petrişan - Aiguier ICALP’21)

- There is at most one monotone distributive law $T\mathbb{E} \to \mathbb{E}T$.
- Existence $\iff T$ functor, unit, multiplication are nearly cartesian $T$ preserves strong epis
Toposes and $\mathfrak{E} \mathfrak{E} \to \mathfrak{E} \mathfrak{E}$

Fact: every topos has a powerset monad $\mathfrak{E}$.

Theorem (G. - Petrişan - Aiguier ICALP’21)

- There is at most one monotone weak distributive law $T\mathfrak{E} \to \mathfrak{E}T$.
- Existence $\iff$ $T$ functor, multiplication are nearly cartesian $T$ preserves strong epis
Toposes and $\exists \exists \to \exists \exists$

Fact: every topos has a powerset monad $\exists$.

Theorem (G. - Petrişan - Aiguier ICALP’21)

- There is at most one monotone weak distributive law $T\exists \to \exists T$.
- Existence $\iff T$ functor, multiplication are nearly cartesian $T$ preserves strong epis

Theorem (G. - Petrişan - Aiguier ICALP’21)

There is a monotone weak distributive law $\exists \exists \to \exists \exists$. In the internal logic,

$$(t : \Omega^{\Omega^X}) \vdash \lambda_X (t) = \{ s : \Omega^X \mid (\forall (x : X), x \in s \to x \in \mu_X(t)) \wedge \forall (u : \Omega^X). u \in t \to \exists (x : X). x \in u \wedge x \in s \}$$

This is a distributive law iff the topos is degenerate.

- Generalisation of the monotone $PP \to PP$ in Set
- Intermediate result before KHaus
Coq proofs for toposes

- Proofs in Coq \equiv proofs in the internal logic
- \( \text{Prop} \equiv \text{subobject classifier} \)

```coq
Definition P X := X \to \text{Prop}.
Definition im [X] [Y] (f : X \to Y) (a : P X) (y : Y) :=
  exists (x : X), a x \land f x = y.
Definition etaP X (x : X) (x' : X) := x = x'.
Definition muP X (t : P (P X)) (x : X) := exists (s : P X), s x \land t s.
```

```coq
Theorem eta_nearly_cartesian :
  (forall X Y (f : X \to Y) (s : P X) (y : Y),
  im f s = etaP Y y \to
  exists (x : X), etaP X x = s \land f x = y) \iff (\text{True} = \text{False}).
```

```coq
Theorem mu_nearly_cartesian :
  (forall X Y (f : X \to Y) (s : P X) (t' : P (P Y)),
  im f s = muP Y t' \to
  exists (t : P (P X)), muP X t = s \land im (im f) t = t').
```

```coq
Theorem monotone_weak_dlaw X :
  (forall (t : P (P X)) (s : P X),
  lambda_m X t s \iff
  (forall x : X, s x \to muP X t x)
  \land (forall u : (P X), t u \to exists (x : X), u x \land s x).
```

- ... and a formalisation of \( \text{No } PP \to PP \) (Klin - Salamanca 18)
The conjecture $RV \rightarrow VR$

Conjecture (Generalised $DP \rightarrow PD$)

There is a monotone weak distributive law $RV \rightarrow VR$. For $m \in RVX$,

$$\lambda_X(m) = \left\{ m' \text{ Radon measure on } X \text{ such that } \forall (C, B) \in VVX \times VX, \bigcup C \subseteq B \Rightarrow m(C) \leq m'(B) \right\}$$

Proof.

- $R$ preserves strong epis ✓
- $R$ is nearly cartesian ✓
- multiplication is nearly cartesian ?
- $\text{Rel}(R)$ preserves continuity ?

Expression of $\lambda$ is obtained via (Edwards 78).
Citations

The slides cite [16, 18, 22, 2, 20, 21, 14, 24, 23, 8, 1, 9, 10, 15, 19, 13, 11, 4, 12, 3, 5, 7, 17, 6]